3

SIMPLE OSCILLATORY SYSTEMS

1. Why belabor the theory of oscillators? A mass point m moves 1-dimensionally subject to the conservative force $F(z) = -U'(z)$:

$$m\ddot{z}(t) = -U'(z)$$

Looking to a graph of the potential (FIGURE 1) we notice that $U(z)$ has a local minimum at $z = a$:

$$U'(a) = 0 \quad \text{and} \quad U''(a) > 0$$

Expanding about that point we have

$$U(z) = U(a) + U'(a)(z-a) + \tfrac{1}{2}U''(a)(z-a)^2 + \tfrac{1}{6}U'''(a)(z-a)^3 + \cdots$$

The constant $U(a)$ makes no contribution to the force, so can be abandoned. The term of order $(z-a)^1$ is actually absent because a is an extremal point. So we have

$$U(z) \approx \tfrac{1}{2}U''(a)(z-a)^2 + \text{higher order terms}$$

So long as the "excursion variable" $x \equiv z - a$ remains small we can, in leading approximation, abandon the higher order terms, and are left with

$$m\ddot{x} = -kx \quad : \quad k \equiv U'(a) > 0 \tag{1}$$

which we recognize to be the equation that describes the motion of a particle that is bound to the origin by an ideal spring.

We study (1) not because much of the universe consists of particles literally attached to springs, but because a great variety of systems can be considered to be **jiggling about points of stable equilibrium**. Language and analytical methods devised to treat (1) and its variants inform the discussion of *all* such systems.

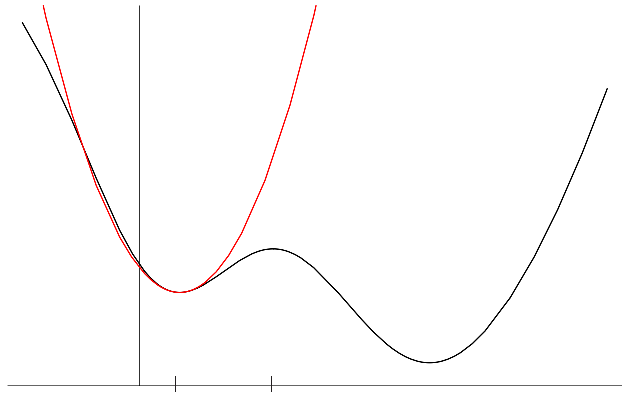

FIGURE 1: *Shown in black is a potential $U(z)$ with a local minimum at $z = a$, a local maximum at $z = b$, a global minimum at $z = c$. Shown in red is the "osculating parabolic potential" that results when one expands $U(z)$ as a Taylor series in powers of $(z - a)$ and drops the terms of order greater than 2.*

PROBLEM 1 : Plot $U(x) = -10 + (x-5)^2 + (x-1)^3 e^{\frac{1}{4}(x+1)^2}$ on the interval $-3 \leqslant x \leqslant 10$. Use `FindRoot` to locate the position a of the local minimum that will be evident on your graph. Superimpose a graph of the function that results when you expand about $x = a$ and drop terms of order higher than two.

Dimensionally $[m] = M^1 L^0 T^0$ and $[k] = M^1 L^0 T^{-2}$ so

$$\omega \equiv \sqrt{k/m} \quad : \quad [\omega] = M^0 L^0 T^{-1}$$

comprises a **natural frequency** that we can expect will figure prominently in the discussion of all such systems. Equation (1) has become $m\ddot{x} = -m\omega^2 x$, or more simply

$$\ddot{x} + \omega^2 x = 0 \tag{2}$$

This equation is, as will emerge, distinguished most importantly (and from most other instances of $m\ddot{x} = F(x)$) by its **linearity**: if $x_1(t)$ and $x_2(t)$ are solutions of (2) then so also are all functions of the form

$$x(t) = c_1 x_1(t) + c_2 x_2(t) \quad : \quad c_1 \text{ and } c_2 \text{ are arbitrary constants}$$

As it happens, acoustics/electrodynamics/quantum mechanics are all linear theories, dominated by **principls of superposition**, and many other theories are usefully studied and applied **in linear approximation**. Lessons learned from study of (2) and its variants pertain to *all* of those subject areas. Time spent studying (2) is therefore certainly not time wasted.

2. Simple solution strategies. We might, as would be probably be our first impulse in more complicated situations, simply ask *Mathematica* for solutions of (2). The command `DSolve[x″[t] + ω²x[t]==0, x[t], t]` supplies

$$x(t) = C_1 \cos \omega t + C_2 \sin \omega t \tag{3.1}$$

which—since the arbitrary constants C_1 and C_2 are in number equal to the order of the differential equation—must be the *general* solution.

> **PROBLEM 2** : Use the `Solve` command to discover the values that must be assigned to C_1 and C_2 in order to achieve
>
> a) the initial conditions
>
> $$x\,(0) = x_0$$
> $$x'(0) = v_0$$
>
> b) the terminal conditions
>
> $$x(t_1) = x_1$$
> $$x(t_2) = x_2$$

Writing $C_1 = A \cos \delta$ and $C_2 = -A \sin \delta$ we find that (3.1) can be written

$$x(t) = A \cos(\omega t + \delta) \tag{3.2}$$

where A refers to the **amplitude** of the particle's oscillatory trajectory, and δ to its (initial) **phase**.

So simple are the solutions (3) that one might simply have *guessed* them. Which would have been fair: *any* differential equation-solving method is fair, however outrageous it might seem (and we, before we are done, will encounter some outrageous ones!), for you can always demonstrate after the fact that your purported solution *is* a solution.

COMPLEX VARIABLE METHODS Look now to this typical linear differential equation with real coefficients

$$a_3\dddot{x} + a_2\ddot{x} + a_1\dot{x} + a_0 x = 0$$

The underlined conditions permit us to regard that equation as the real part of an identical equation in the complex variable $z = x + iy$. Why bother to adopt such a viewpoint? Because it permits us to entertain the guess/hypothesis/conjecture/Ansatz that the equation possesses solutions of the form

$$z(t) = Ze^{i\nu t}$$

No function is easier to differentiate: immediately

$$Z\big\{-a_3 i\nu^3 - a_2\nu^2 + a_1 i\nu + a_0\big\}e^{i\nu t} = 0$$

which requires that ν be one or another of the roots $\{\nu_1, \nu_2, \nu_3\}$ of the polynomial

$$-a_3 i\nu^3 - a_2\nu^2 + a_1 i\nu + a_0 = 0$$

We are led thus (by superposition; *i.e.*, by linearity) to solutions of the form

$$x(t) = \Re\big[Z_1 e^{i\nu_1 t} + Z_2 e^{i\nu_2 t} + Z_3 e^{i\nu_3 t}\big]$$

where $\Re$ signifies "real part of." Returning in the light of these remarks to (2), we have

$$\ddot{z} + \omega^2 z = 0$$

and from the Ansatz $z(t) = Ze^{i\nu t}$ obtain

$$Z\big\{-\nu^2 + \omega^2\big\}e^{i\nu t} = 0$$

giving $\nu = \pm\omega$, whence

$$z(t) = Z_1 e^{+i\omega t} + Z_2 e^{-i\omega t}$$

Write $Z_1 = A_1 + iB_1$, $Z_2 = A_2 + iB_2$ and use Euler's identity (1740)

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{4}$$

to obtain

$$\begin{aligned} z(t) = &\ \big[(A_1 + A_2)\cos\omega t - (B_1 - B_2)\sin\omega t\big] \\ &+ i\big[(B_1 + B_2)\cos\omega t + (A_1 - A_2)\sin\omega t\big] \end{aligned}$$

The real part of this result differs only notationally from (3.1).

PROBLEM 3 : a) Suppose the complex variable $z = x + iy$ satisfies a linear differential equation

$$a_2\ddot{z} + (a_1 + ib_1)\dot{z} + a_0 z = 0$$

in which we have *sacrificed reality of the coefficients*. Show that y has now joined x in the real part of that equation, and that x has joined y in the imaginary part; *i.e.*, that x and y have become <u>coupled</u>: one cannot solve for one without simultaneously solving for the other.

b) Suppose that z satisfies an equation

$$a_2\ddot{z} + a_1\dot{z}\dot{z} + a_0 z = 0$$

in which we have *sacrificed linearity*. Show that x and y have again become coupled—now for a different reason. You may find the command `ComplexExpand` useful in this connection.

PHASE PLANE METHODS The linear momentum of our mass point is given by $p = m\dot{x}$, and its introduction permits (2) to be written $\dot{p} + m\omega^2 x = 0$. So we have—jointly equivalent to the single 2^{nd}-order differential equation

$$\ddot{x} + \omega^2 x = 0$$

—this coupled pair of 1^{st}-order differential equations:

$$\left.\begin{aligned} \dot{x} &= +p/m \\ \dot{p} &= -m\omega^2 x \end{aligned}\right\} \tag{5}$$

In matrix notation

$$\tfrac{d}{dt}\begin{pmatrix} x \\ p \end{pmatrix} = \mathbb{M}\begin{pmatrix} x \\ p \end{pmatrix} \quad \text{with} \quad \mathbb{M} \equiv \begin{pmatrix} 0 & 1/m \\ -m\omega^2 & 0 \end{pmatrix}$$

Evidently

$$\begin{pmatrix} x \\ p \end{pmatrix}_t = e^{\mathbb{M}t}\begin{pmatrix} x \\ p \end{pmatrix}_0 \tag{6}$$

and *Mathematica*'s `MatrixExp` command supplies

$$e^{\mathbb{M}t} = \begin{pmatrix} \cos\omega t & (m\omega)^{-1}\sin\omega t \\ -(m\omega)^{+1}\sin\omega t & \cos\omega t \end{pmatrix}$$

giving

$$x(t) = x_0\cos\omega t + (p_0/m\omega)\sin\omega t \tag{7.1}$$

$$\begin{aligned} p(t) &= -(x_0 m\omega)\sin\omega t + p_0\cos\omega t \\ &= m\dot{x}(t) \end{aligned} \tag{7.2}$$

Elimination of t between those equations would supply a description of the curve $\mathcal{C}$ that the moving phase point $\{x(t), p(t)\}$ traces in 2-dimensional **phase space**.[1] To that end, we use `Solve` to obtain

$$\begin{aligned} \frac{pp_0 + m^2\omega^2 x x_0}{p_0^2 + m^2\omega^2 x_0^2} &= \cos\omega t \\ \frac{m\omega x p_0 - m\omega p x_0}{p_0^2 + m^2\omega^2 x_0^2} &= \sin\omega t \end{aligned}$$

Square, add and simplify: get

$$\frac{p^2 + m^2\omega^2 x^2}{p_0^2 + m^2\omega^2 x_0^2} = 1$$

But $\frac{1}{2m}(p^2 + m^2\omega^2 x^2) = E$ is just the conserved **total energy** of our oscillator,

[1] "Phase" is an overworked word. This usage has nothing to do with the usage encountered on page 3.

so we can, if we wish (recall that the spring conatant $k = m\omega^2$), write

$$\frac{p^2 + m^2\omega^2 x^2}{2mE} = \frac{p^2}{(\sqrt{2mE})^2} + \frac{x^2}{(\sqrt{2E/k})^2} = 1 \tag{8}$$

But (8) describes an **ellipse** for which

- amplitude $A \equiv x_{\max} = \sqrt{2E/k} \Longrightarrow E = \frac{1}{2}kA^2$ (the energy is all potential, stored in the spring);
- $P \equiv p_{\max} = \sqrt{2mE} \Longrightarrow E = \frac{1}{2m}P^2$ (the energy is all kinetic).

It follows in particular that the **area** of the ellipse is given by

$$\text{area} = \pi AP = 2\pi E/\omega \tag{9}$$

REMARK: Quantum mechanics was born (1900) when Planck found himself forced to conjecture that the only oscillator motions achievable in Nature are those for which

$$\text{area} = nh \quad : \quad n = 1, 2, 3, \ldots$$

where h is a new natural constant, with the physical dimensions

$$[h] = [xp] = M^1 L^2 T^{-1}$$

of "action." Thus was Planck led from (9) to this formula

$$E_n = n\hbar\omega \quad : \quad \hbar \equiv h/2\pi$$

for the "allowed energies" of a quantum oscillator.

The expression that describes oscillator energy is easily (and, as will emerge, very usefully) factored:

$$\begin{aligned} E &= \tfrac{1}{2m}(p^2 + m^2\omega^2 x^2) \\ &= \mathcal{E} \cdot \frac{m\omega x - ip}{\sqrt{2m\mathcal{E}}} \cdot \frac{m\omega x + ip}{\sqrt{2m\mathcal{E}}} \end{aligned} \tag{10}$$

Here $\mathcal{E}$ is a dimensioned constant ($[\mathcal{E}] =$ energy) of arbitrary value, introduced to insure that the subsequent factors are dimensionless.[2] Algebraic inversion (use `Solve`) of

$$\left.\begin{aligned} a &= \frac{m\omega x - ip}{\sqrt{2m\mathcal{E}}} \\ a^* &= \frac{m\omega x + ip}{\sqrt{2m\mathcal{E}}} \end{aligned}\right\} \tag{11}$$

[2] From m and ω—the only physical parameters available to the classical theory of oscillators—it is not possible to construct such a constant (it must be pulled out of thin air), but in quantum theory it becomes natural to set $\mathcal{E} = \hbar\omega$.

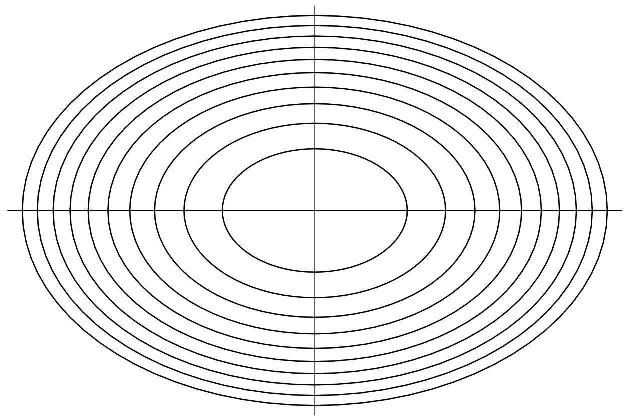

FIGURE 2: *Ellipses inscribed on the phase plane by oscillators having identical values of m and k (equivalently: of m and ω) but ascending energies, which have been graded by Planck's rule $E = E_0 n : n = 1, 2, 3, \ldots$ When n becomes very large the ellipses become too finely spaced to be individually resolved: we have at that point entered the "classical world."*

gives

$$\begin{aligned} x &= \qquad \sqrt{\mathcal{E}/2m}\,(a + a^*)/\omega \\ p &= im^2\sqrt{\mathcal{E}/2m}\,(a - a^*) \end{aligned}$$

in which notation (5) becomes, after some simplification,

$$\begin{aligned} (\dot{a} + \dot{a}^*) &= i\omega(a - a^*) \\ (\dot{a} - \dot{a}^*) &= i\omega(a + a^*) \end{aligned}$$

Adding/subtracting those equations we obtain finally

$$\dot{a} = i\omega a \tag{12}$$

and (redundantly) its complex comjugate. Immediately

$$a(t) = a_0\, e^{i\omega t} \tag{13}$$

which traces (uniformly) not an ellipse but a **circle on the complex a-plane**. It is by deformation of the physical x and p variables

$$\left.\begin{aligned} x \longmapsto \varkappa &= \frac{m\omega}{\sqrt{2m\mathcal{E}}}\,x \\ p \longmapsto \wp &= \frac{1}{\sqrt{2m\mathcal{E}}}\,p \end{aligned}\right\} \quad : \quad a = \varkappa - i\wp$$

that what was formerly an ellipse has been rendered circular, and energy has become proportional to the squared radius of the circle: $E = \mathcal{E}\cdot a^*a = \mathcal{E}(\varkappa^2 + \wp^2)$

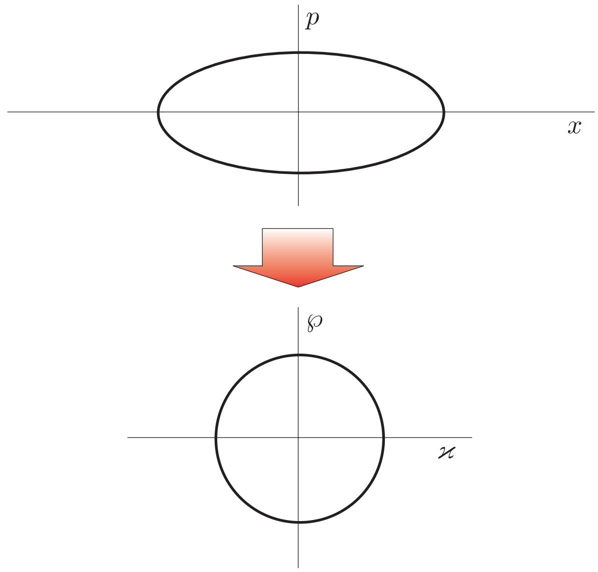

FIGURE 3: *Representation of the deformation of ellipse into circle that is accomplished by introduction of the complex variable a. The top figure is inscribed on physical phase space, the bottom figure on the dimensionless complex plane. Energy is proportional through a dimensioned factor to the squared radius of the circle.*

The trickery sketched above is as consequential as it is elegant. The variables a^* and a acquire in the quantum theory of oscillators the status of "step up" and "step down ladder operators," devices that sprang from the imagination of P. A. M. Dirac.[3] In quantum field theory they (or objects formally identical to them) become the operators that represent the creation and anihilation of particles.

PROBLEM 4 : Supposing $x(t)$ to have been presented in the form (3.2), construct a description of $p(t)$. Setting $\delta = 0$, $A = \omega = 1$ and assigning several illustrative values to m, use `ParametricPlot` to graph $\big\{x(t), p(t)\big\}$ as t ranges from 0 to 6. Be sure to include the stipulation `AspectRatio→Automatic`.

PROBLEM 5 : Use (5) to construct a demonstration that energy $E = \frac{1}{2m}p^2 + \frac{1}{2}kx^2$ is conserved.

PROBLEM 6 : Hamilton's **canonical equations of motion** read

$$\dot{x} = +\tfrac{\partial}{\partial p}H(x,p), \quad \dot{p} = -\tfrac{\partial}{\partial x}H(x,p)$$

What "Hamiltonian" $H(x,p)$ would serve to give back (5).

[3] See, for example, David Griffiths, *Introduction to Quantum Mechanics* (2^{nd} edition 2005), §2.3.1.

3. A simple model of the effect of energy dissipation. All oscillators—certainly all macroscopic mechanical oscillators[4]—come evenually to rest, losing their initial store of energy to (say) the production of sound, or of light, or of frictive heat. In place of

$$m\ddot{x} = -kx$$

one should expect generally to have to write

$$m\ddot{x} = -kx + F_{\text{dissipation}}$$

We expect $F_{\text{dissipation}}$ to depend complicatedly upon the phenomenological details of the dissipation mechanism as hand, but in all cases to

- to be *directed opposite to the momentary direction of motion* (as given by the sign of $\dot{x}$);
- to vanish when the particle is at rest.

The only way *consistent with linearity* to realize those conditions is to set $F_{\text{dissipation}} \sim -\dot{x}$; *i.e.*, to write

$$F_{\text{dissipation}} = -D\dot{x}$$

and it is to this simple model that we will restrict our attention.[5] Whether or

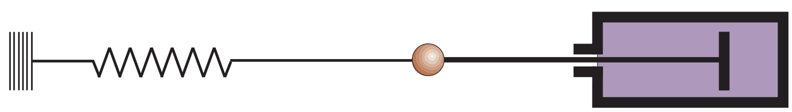

FIGURE 3: *Particle connected to a spring and to a "dashpot," which subjects the particle to a $\dot{x}$-dependent viscous force.*

not we adopt the "dashpot" language of engineers, our assignment is to solve and to discuss the physical implications of the linear differential equation

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0 \tag{14}$$

where $\gamma = \frac{1}{2}D$ is a positive constant and, for reasons that will emerge, we have agreed to write ω_0 where formerly we wrote simply ω.

`DSolve` supplies

$$x(t) = C_1 e^{\,t\left(-\gamma-\sqrt{\gamma^2-\omega_0^2}\right)} + C_2 e^{\,t\left(-\gamma+\sqrt{\gamma^2-\omega_0^2}\right)} \tag{15}$$

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[4] Quantum mechanics provides avenues of escape, of a sort.

[5] This we do in full recognition of the fact that we may encounter situations in which it would be more appropriate to write (say)

$$F_{\text{dissipation}} = -f_1\dot{x} - f_3\dot{x}^3 - f_5\dot{x}^5 - \cdots$$

or to consider one of the many still more complicated possibilities.

which is also very easily obtained by hand, using the complex variable method described on page 4. Bring to

$$\ddot{z} + 2\gamma\dot{z} + \omega_0^2 z = 0$$

the Ansatz $z(t) = Ze^{i\nu t}$, obtain $Z\{-\nu^2 + 2i\gamma\nu + \omega_0^2\}e^{i\nu t} = 0$ which gives

$$\nu = i\big(\gamma \pm \sqrt{\gamma^2 - \omega_0^2}\big)$$

whence

$$z(t) = Z_1 e^{t\left(-\gamma - \sqrt{\gamma^2-\omega_0^2}\right)} + Z_2 e^{t\left(-\gamma + \sqrt{\gamma^2-\omega_0^2}\right)}$$

Note the sense in which we recover $z(t) = Z_1 e^{+i\omega_0 t} + Z_2 e^{-i\omega_0 t}$ in the limit $\gamma \downarrow 0$.

UNDERDAMPED CASE: $\omega^2 \equiv \omega_0^2 - \gamma^2 > 0$ Arguing as on page 3 we are led from (15) to

$$x(t) = e^{-\gamma t}\{\alpha \cos\omega t + \beta \sin\omega t\} \tag{16.1}$$

$$= Ae^{-\gamma t} \cdot \cos(\omega t + \delta) \tag{16.2}$$

The particle oscillates with diminished frequency $\omega < \omega_0$ and exponentially dying amplitude, as illustrated in FIGURE 5. Bringing $p(t) = m\dot{x}(t)$ into play we get FIGURE 6.

OVERDAMPED CASE: $\omega_0^2 - \gamma^2 < 0$ The exponents in (15) are now both real: the damping is now so strong as to prevent "oscillatory overshoot." Writing

$$\Gamma_\pm \equiv \gamma \pm \sqrt{\gamma^2 - \omega_0^2} \tag{17.1}$$

and noticing that $0 < \Gamma_- < \Gamma_+$, we have

$$x(t) = A_{\text{fast}} e^{-\Gamma_+ t} + A_{\text{slow}} e^{-\Gamma_- t} \tag{17.2}$$

Only after the fast component has died does the relaxation to equilibrium become truly "exponential" (see FIGURE 7). Of course, the values of $A_{\text{fast/slow}}$ are set by the initial conditions, and if those have been chosen so as to achieve $A_{\text{slow}} = 0$ then the fast compoonent will predominate.

CRITICALLY DAMPED CASE: $\omega_0^2 - \gamma^2 = 0$ In this case (15) assumes the form

$$x(t) = Ce^{-\gamma t}$$

which is indeed a *particular* solution of $\ddot{x} + 2\gamma\dot{x} + \gamma^2 x = 0$ but, since it contains only a single adjustable constant, cannot be the *general* solution. Turning again to *Mathematica*, we obtain in this instance

$$x(t) = (C_1 + C_2 t)e^{-\gamma t} \tag{18}$$

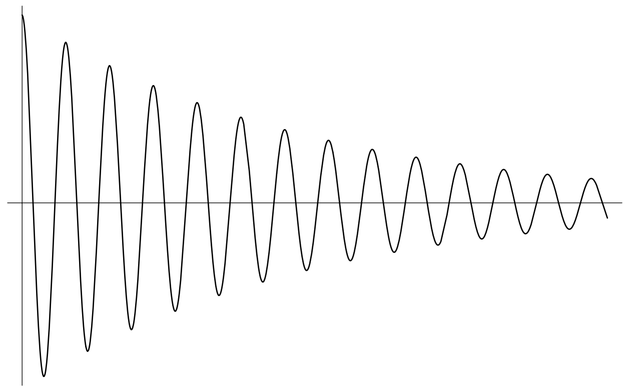

FIGURE 5: *Graph of the motion of an underdamped oscillator, based upon an instance of* (16).

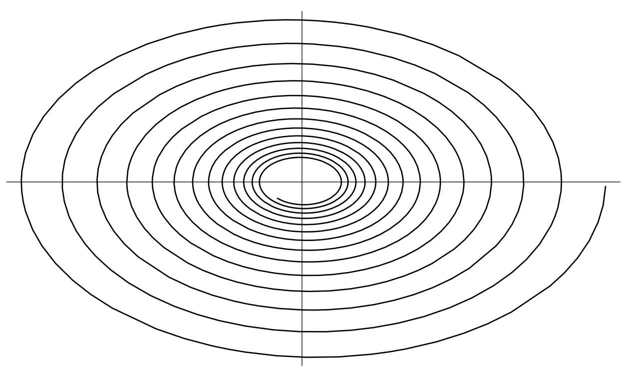

FIGURE 6: *The same motion displayed in phase space, after m has been assigned a typical value.*

PROBLEM 7: a) Working from (16.1), and writing x_0 and v_0 to denote initial position and velocity, show that

$$\alpha = x_0 \quad \text{and} \quad \beta = \frac{v_0 + \gamma x_0}{\omega}$$

b) Show that in the limit $\omega_0^2 \downarrow \gamma^2$ one obtains an equation of the form (18).

By way of application: it is difficult to obtain accurate results (or *any* result quickly) with a balance or electrical meter (of d'Arsonval's classic pre-digital design) if the pointer oscillates back and forth about the true value of the variable being measured (FIGURE 5). For that reason, engineers turn up the value of γ until the device is brought to the critical edge of the overdamped

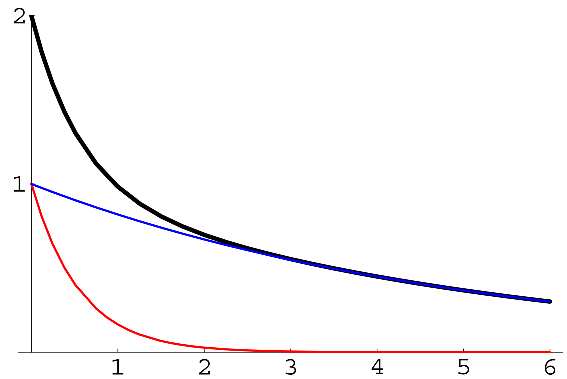

FIGURE 7: *Overdamped motion of an oscillator. In constructing the figure, which is based upon* (17)*, I have set* $A_{\text{fast}} = A_{\text{slow}} = \gamma = 1$ *and* $\sqrt{\gamma^2 - \omega_0^2} = 0.2$. *Only after the fast component* (*shown here in red*) *has effectively died can the motion properly be said to "relax exponentially," with characteristic time* $\tau = (\gamma - \sqrt{\gamma^2 - \omega_0^2})^{-1}$.

regime. From $\gamma > \gamma - \sqrt{\gamma^2 - \omega_0^2}$ it follows that an instrument thus carefully tuned provides the least possible wait-time before a reading can be taken.[6]

4. The "Q" of an underdamped oscillator. As—under the control of γ—a damped oscillator winds down its energy diminishes (being exchanged with other parts of the universe, usually in the form of heat) until ultimately it is depleted. We look to the details of that process, as they relate specifically to *under*damped oscillators. Working from (16.2) we have[7]

$$
\begin{aligned}
2E(t)/m = \dot{x}^2 + \omega_0^2 x^2 &= A^2 e^{-2\gamma t}\Big\{\big[\gamma\cos\omega t + \omega\sin\omega t\big]^2 + \omega_0^2\cos^2\omega t\Big\}\\
&= A^2 e^{-2\gamma t}\Big\{(\gamma\cos\omega t)^2 + 2\gamma\omega\cos\omega\sin\omega t + (\omega_0^2 - \gamma^2)\sin^2\omega t + \omega_0^2\cos^2\omega t\Big\}\\
&= A^2 e^{-2\gamma t}\Big\{\omega_0^2 + \gamma\big[\gamma\cos 2\omega t + \omega\sin 2\omega t\big]\Big\}
\end{aligned}
$$

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[6] Jacques-Arsène d'Arsonval (1851–1940) was a physicist who deserves to be much better known. For basic information go to

http://chem.ch.huji.ac.il/ eugeniik/history/arsonval.html

The fact that he was not born until nearly twenty years after Micahel Faraday performed his most celebrated electrical experiments, and did not invent the "d'Arsonval meter movement" until 1882, makes me wonder what kind of "galvanometer" Faraday might possibly have used. This is a question for which I have yet to discover an answer.

[7] To reduce irrelevant notational clutter I set $\delta = 0$.

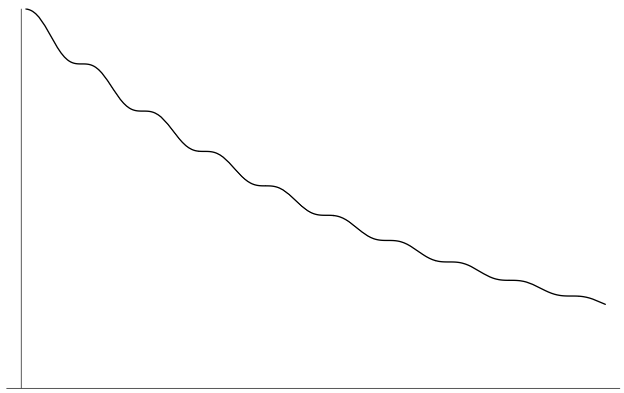


FIGURE 8: *Figure based upon* (19), *showing energy rippling* (*at doubled frequency*) *exponentially* (*at doubled rate*) *to extinction.*

Using $\omega_0^2 = \omega^2 + \gamma^2$ as an invitation to write

$$\omega = \omega_0 \cos\alpha$$
$$\gamma = \omega_0 \sin\alpha$$

we obtain at last

$$E(t) = \tfrac{1}{2}m\omega_0^2 A e^{-2\gamma t}\cdot\big[1 + (\gamma/\omega_0)\sin(2\omega t + \alpha)\big] \tag{19}$$

So energy does not die exponentially: it dies as an exponential modulated by a factor that oscillates between $\omega_0 + \gamma$ and $\omega_0 - \gamma$ (see the figure). The energy lost per period is given by

$$\begin{aligned}\Delta E(t) &= E(t) - E(t+\tau)\\ &= \tfrac{1}{2}m\omega_0^2 A^2 e^{-2\gamma t}\cdot\big[1 - e^{-2\gamma\tau}\big]\big[1 + (\gamma/\omega_0)\sin(2\omega t + \alpha)\big]\end{aligned} \tag{20.1}$$

while than value of $E(t)$—average over one cycle—is (or so *Mathematica* informs us)

$$\begin{aligned}\langle E(t)\rangle &= \tfrac{1}{\tau}\int_t^{t+\tau} E(t')\,dt'\\ &= \tfrac{1}{2}m\omega_0^2 A^2 e^{-2\gamma t}\cdot\big[1 - e^{-2\gamma\tau}\big]\tfrac{1}{2\gamma\tau}\big[1 + (\gamma/\omega_0)^2\sin(2\omega t + \alpha)\big]\end{aligned} \tag{20.2}$$

The dimensionless ratio

$$\frac{\text{mean stored energy during a cycle}}{\text{energy lost during that cycle}}$$

being of obvious significance, we use (20) to construct

$$Q(t) \equiv 2\pi\frac{\langle E(t)\rangle}{\Delta E(t)} = \frac{\pi}{\gamma\tau}\cdot\left\{\frac{\big[1 + (\gamma/\omega_0)^2\sin(2\omega t + \alpha)\big]}{\big[1 + (\gamma/\omega_0)\ \ \sin(2\omega t + \alpha)\big]}\right\}$$

where most of the t-dependence has dropped away. That which remains is due to "ripple terms," terms that sense at what point in a cycle we started our clock. And those drop away if we assume the damping to be slight ($\gamma/\omega_0 \ll 1$) or—alternatively—we average over a cycle, constructing $Q \equiv \langle Q(t) \rangle$. Thus does it come about that when engineers speak of the "Q-value" of a weakly damped oscillator they refer to

$$\begin{aligned} Q \equiv \frac{\pi}{\gamma\tau} &= \frac{\omega}{2\gamma} \\ &\approx \frac{\omega_0}{2\gamma} \quad \text{if the damping is weak: } \gamma \ll \omega_0 \end{aligned}$$

Oscillators with high Q-values lose energy slowly: they "ring" for a long time; *i.e.*, for many cycles. Q is susceptible to easy measurrement, and provides a direct estimate of the value of γ.

> **PROBLEM 8**: The intensity of the sound produced by a 440 Hz tuning fork drops by a factor of $\frac{1}{2}$ every 1.5 seconds. What is the Q-value of the tuning fork?

5. Response of a damped oscillator to externally impressed forces. Suppose now that our mass point m—subjected already to a restoring (or "spring") force and a damping force—is subjected also to a time-dependent impressed force $F(t)$. Our homogeneous differential equation of motion (14)—of which I present here a newly-number copy

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0 \tag{21.1}$$

—is replaced now by the inhomogeneous equation

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x \overset{\downarrow}{=} S(t) \tag{21.2}$$

It will be my non-standard practice to call $S(t) \equiv \frac{1}{m}F(t)$ the "stimulus." Our assignment is to construct and interpret the solutions of (21.2).

It is important to notice that if $x_1(t)$ is any solution (21.2) with stimulus $S_1(t)$, and $x_2(t)$ any solution with stimulus $S_2(t)$, then $x(t) = c_1x_1(t) + c_2x_2(t)$ will be a particular solution when the stimulus is $S(t) = c_1S_1(t) + c_2S_2(t)$. From this simple fact it follows that if $x_s(t)$ is any *particular solution of the inhomogeneous equation* (21.2), and $x_o(t)$ is the *general solution of the associated homogeneous equation* (21.1), then

$$x(t) = x_o(t) + x_s(t) \tag{22}$$

will comprise the *<u>general</u> solution of the inhomogeneous equation*. In practice it is usually most convenient to arrange to have $x_s(0) = \dot{x}_s(0) = 0$ and to pass on to $x_o(t)$ the responsibility for assuring that $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$. In the presence of damping all functions of the type $x_o(t)$ die in characteristic time $\tau = 1/\gamma$, as we have seen. And they carry with them to the grave all recollection of the initial conditions! Equation (22) can therefore be said to possess the structure

$$x(t) = \text{transcient} + \text{stimulated}$$

The theory of transcient functions $x_o(t)$ is already under our belts, so it is upon the stimulated functions $x_s(t)$ that we will concentrate. We look first to the important but relatively simple case of harmonic stimulation, then to the case of arbitrary stimulation.

HARMONIC STIMULATION Assuming $S(t)$ to be harmonic

$$S(t) = \text{real part of } Se^{i\nu t}$$

we embrace the familiar Ansatz $x_\nu(t) = X(\nu)e^{i\nu t}$ and from (21.2) obtain

$$\begin{aligned}
\big[-\nu^2 + 2i\gamma\nu + \omega_0^2\big]X(\nu)\,e^{i\nu t} &= Se^{i\nu t}\\
&\downarrow\\
X(\nu) &= \frac{1}{-\nu^2 + 2i\gamma\nu + \omega_0^2}S\\
&= \left\{\frac{\omega_0^2 - \nu^2}{(\omega_0^2 - \nu^2)^2 + 4\gamma^2\nu^2} - i\frac{2\gamma\nu}{(\omega_0^2 - \nu^2)^2 + 4\gamma^2\nu^2}\right\}S\\
&= A(\nu)e^{-i\delta(\nu)}
\end{aligned} \tag{23}$$

with

$$\left.\begin{aligned}
A(\nu) &= \frac{1}{\sqrt{(\omega_0^2 - \nu^2)^2 + 4\gamma^2\nu^2}}\cdot S\\
\delta(\nu) &= \arctan\left[\frac{2\gamma\nu}{\omega_0^2 - \nu^2}\right]
\end{aligned}\right\} \tag{24}$$

REMARK: We have made use here of these elementary properties of complex numbers: if $z = x + iy$ then

$$\frac{1}{z} = \frac{1}{x+iy}\cdot\frac{x-iy}{x-iy} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

$$\begin{aligned}
z &= \sqrt{x^2+y^2}\cdot e^{+i\arctan(y/x)}\\
\frac{1}{z} &= \frac{1}{\sqrt{x^2+y^2}}\cdot e^{-i\arctan(y/x)}
\end{aligned}$$

PROBLEM 9: Instead of looking for the response of our damped oscillatorto harmonic stimulation one might—more efficiently, as will emerge—look for the simulation that produces a harmonic response. To demonstrate the point, insert $x(t) = A\sin\nu t$ into (21.2) and minipulate the resulting $S(t)$ until it assumes the form $S\sin(\nu t + \delta)$. Notice that you could, by this means, discover the stimulus $S(t)$ that produces *any* preassigned response!

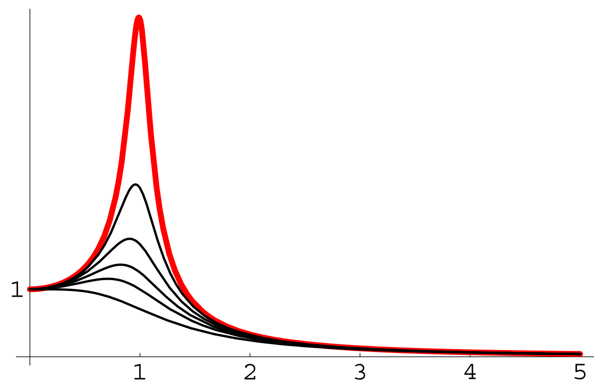


FIGURE 9: *In constructing these graphs of $A(\nu)$, which are based upon (24), I have set $S = \omega_0 = 1$ and assigned to γ the successive values $\{0.1, 0.2, 0.3, 0.4$ and $0.707107 = 1/\sqrt{2}\}$. The heavy red curve arises when the damping is weakest: $\gamma = 0.1$. As damping increases the frequency at which $A(\nu)$ is maximal is given by*

$$\nu_{\max} = \sqrt{\omega_0^2 - 2\gamma^2}$$

which shifts downward from ω_0 as γ increases, until at $\gamma = \omega_0/\sqrt{2}$ one has $\nu_{\max} = 0$.

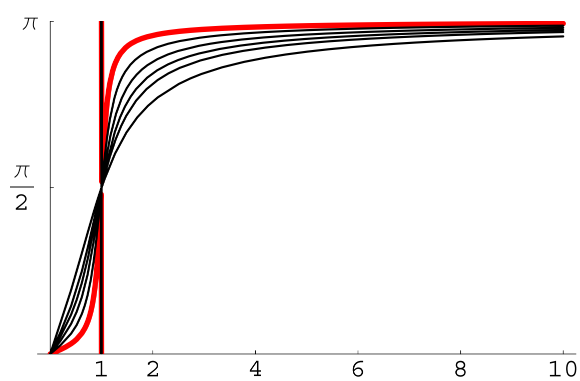


FIGURE 10: *Graphs of $\delta(\nu)$, which describes how much the phase of the response $x_s(t)$ lags behind that of the harmonic stimulus $S(t)$. The values assigned to ω_0 and γ are the same as those used in the preceding figure. If the damping is very weak the phase lag jumps abruptly from 0 to π as ν ascends upward through $\nu = \omega_0$.*

Looking to Figures 9 & 10 we see that (24) speaks of a **resonance** when the frequency of the stimulus lies in the neighborhood of the natural frequency ω_0 of the oscillator. This is familiar news to people who push swings (push at the natural frequency of the lightly damped swing, either in phase or 180^0 out of phase), but it would be bad news for a designer of audio speakers if one narrow range of the frequencies present in a complex signal was enhanced, while other frequencies were repressed. Such an engineer would want to arrange that the maximal response is broad enough to embrace the audio spectrum. Or, short of that, to use several speakers (bass, mid-range, treble) that collectively do the job.

The important point is that a harmonically stimulated oscillator oscillates *at the frequency* ν *of the stimulus*—enthusiastically or reluctantly according as ν is near to or far from the natural frequency ω_0 of the oscillator.

The **energy** of a harmonically stimulated oscillator can be described

$$\begin{aligned}E &= \tfrac{1}{2}m\nu^2A^2(\nu)\\ &= \tfrac{1}{2}mS^2\frac{\nu^2}{(\nu-\omega_0)^2(\nu+\omega_0)^2+4\gamma^2\nu^2}\end{aligned}$$

which for $\nu\sim\omega_0$ can be approximated (the trick: set $\nu+\omega_0=2\nu$)

$$= \tfrac{1}{8}mS^2\frac{1}{(\nu-\omega_0)^2+\gamma^2}$$

This result is more usefully written

$$= \frac{\pi m}{4\Gamma}\cdot\underbrace{\frac{\Gamma}{2\pi}\frac{1}{(\nu-\omega_0)^2+(\Gamma/2)^2}}_{\equiv\,\mathcal{L}(\nu;\omega_0,\Gamma)}$$

—the reason being that $\mathcal{L}(\nu;\omega_0,\Gamma)$ is the famous **Lorentz distribution function**[8] that describes spectral line shape and much else. It has the form shown in Figure 11, and these notable properties:

- $\displaystyle\int_{-\infty}^{+\infty}\mathcal{L}(\nu;\omega_0,\Gamma)\,d\nu = 1 \quad : \quad \text{all } \omega_0, \text{ all } \Gamma$
- is maximal (centered) at $\nu=\omega_0$
- drops to half-max at $\nu=\omega_0\pm\frac{1}{2}\Gamma$

The mean $\langle\nu\rangle\equiv\int\nu\,\mathcal{L}(\nu;\omega_0,\Gamma)\,d\nu$ and centered higher moments $\langle(\nu-\omega_0)^n\rangle$ of all orders $n=2,3,4,\ldots$ are *undefined*: the Lorentz distribution is, as Richard Crandall has remarked, "too fat."

[8] Go to http://en.wikipedia.org/wiki/Cauchy_distribution for discussion of some of the remarkable propeties of the Cauchy-Lorentz distribution.

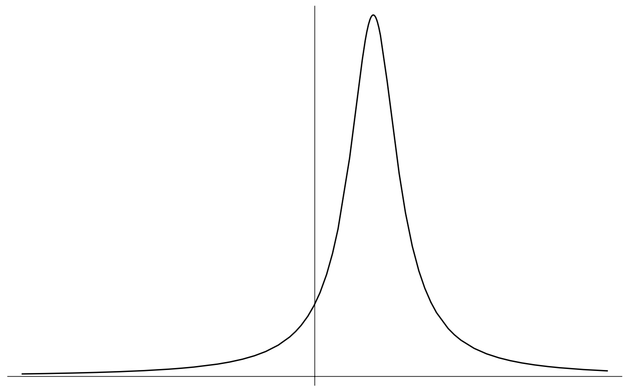


FIGURE 11: *The Lorentz distribution $\mathcal{L}(\nu;\omega_0,\Gamma)$ looks very like the Gaussian (or "normal") distribution, except that its wings are wider—so wide that none of its moments are defined. Easily,*

$$\mathcal{L}_{\max} = \frac{2}{\pi\Gamma}$$

where Γ refers to the width at half maximum.

So we encounter again a point noted already near the top of the preceeding page: a weakly damped oscillator will most willingly drink energy if it is harmonically stimulated at near resonance: $\nu \sim \omega_0$.

We observe finally that from $\Gamma \equiv 2\gamma$ and the $Q \approx \omega_0/2\gamma$ encountered on page 14 it follows that

$$\omega_0/\Gamma = Q \text{ of the \textit{unstimulated} oscillator}$$

ANHARMONIC STIMULATION Demonstrably,[9] the function

$$x(t) = \tfrac{1}{\omega}\int_0^t e^{-\gamma(t-u)}\sin\omega(t-u)S(u)\,du \tag{25}$$

$$\omega \equiv \sqrt{\omega_0^2-\gamma^2} \quad : \quad \text{underdamped case } \gamma < \omega_0$$

satisfies

$$\ddot{x}(t) + 2\gamma\dot{x}(t) + \omega_0^2 x(t) = S(t), \quad x(0)=0 \quad \text{and} \quad \dot{x}(0) = 0$$

I turn now to discussion of a method—there exist others—by which (25) (together with is critically damped and overdamped siblings) can be *derived*, after which I will take up the matter of its interpretation.

[9] **PROBLEM 10**: Write out the demonstration. *Mathematica* is good at evaluating $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x$ but can't seem to manage the simplification, which you will have to do by hand.

Let (21.2) be written

$$\big[D^2 + 2\gamma D + \omega_0^2\big]x(t) = S(t) \quad \text{with} \quad D \equiv \tfrac{d}{dt} \tag{26}$$

I will show that it is not at all outrageous to write

$$\downarrow$$
$$x(t) = \frac{1}{D^2 + 2\gamma D + \omega_0^2} S(t) + x_o(t)$$

and in the course of the argument will demonstrate **the power of good notation and the utility of creative "symbol play."** It is by way of preparation that we look first to the simpler problem

$$\big[D + a\big]x(t) = f(t)$$

We expect to be able to write

$$x(t) = \big[D + a\big]^{-1} f(t) + x_o(t)$$

where $x_o(t)$ satisfies $$\big[D + a\big]x_o(t) = 0$$

But what can be the meaning of $\big[D + a\big]^{-1}$?

> **PROBLEM 11**: Show, by looking to the evaluation of $De^{at}F(t)$, that the **shift rule**
>
> $$D + a = e^{-at}D\,e^{at}$$
>
> is valid as an <u>operator identity</u> (in the sense that the left and right sides yield the same result when applied to any $F(t)$).

Immediately

$$\big[D + a\big]^n = e^{-at}D^n\,e^{at} \quad : \quad n = 0, 1, 2, 3, \ldots$$

What could be more natural, therefore, than to write

$$\big[D + a\big]^{-1} = e^{-at}D^{-1}\,e^{at}$$

and—drawing upon the "fundamental theorem of the calculus"—to

$$\text{interpret } D^{-1} \text{ to mean } \textstyle\int^t$$

The suggestion—and it's hardly more than that—is that we should

$$\text{interpret } \big[D + a\big]^{-1} f(t) \text{ to mean } e^{-at}\int_{t_0}^t e^{au} f(u)\,du$$

PROBLEM 12: You find all this too informal/sloppy to be plausible? I remarked already on page 3 that any differential-equation-solving procedure, no matter how outrageous it may appear, is "fair" since any purported result can always be checked. Proceeding in that spirit, demonstrate that (whatever the value assigned to t_0)

$$x(t) = e^{-at}\int_{t_0}^{t} e^{au} f(u)\,du$$

is in fact a solution of $\big[D+a\big]x(t) = f(t)$. Also evaluate $x(t_0)$, $\dot{x}(t_0)$.

Returning now to (26), we note that (in a notation most natural to the underdamped case: $\gamma^2 < \omega_0^2$)

$$D^2 + 2\gamma D + \omega_0^2 = \big[D - i\Omega_-\big]\big[D - i\Omega_-\big]$$

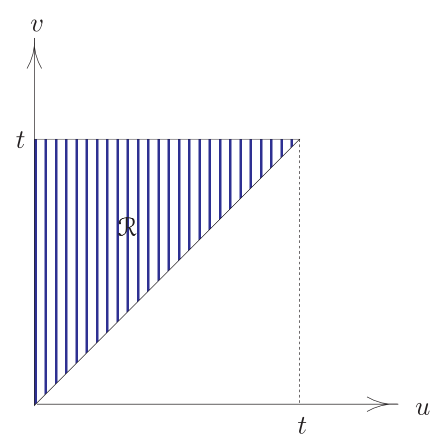
where $\Omega_\pm \equiv \pm\omega + i\gamma$ with $\omega \equiv \sqrt{\omega_0^2 - \gamma^2}$. The evident implication is that

$$\begin{aligned} x(t) &= \big[D - i\Omega_+\big]\cdot e^{i\Omega_- t}\int_0^t e^{-i\Omega_- u} S(u)\,du \\ &= e^{i\Omega_+ t}\int_0^t e^{-i\Omega_+ v}\bigg\{e^{i\Omega_- v}\int_0^v e^{-i\Omega_- u} S(u)\,du\bigg\}\,dv \end{aligned}$$

describes (as one could verify by computation) a particular solution of (26), namely the solution with $x(0) = \dot{x}(0) = 0$. This result, though terrifying on its face, admits of fairly dramatic simplification. We have

$$= e^{-\gamma + i\omega}\iint_{\mathcal{R}} e^{-2i\omega v} e^{(\gamma + i\omega)u} S(u)\,dudv$$

where $0 \leqslant u \leqslant v \leqslant t$ entails that $\iint$ ranges over the region shown in the following figure:



The u-integral cannot be performed until $S(u)$ has been specified, so we do the v-integral first, "while we are waiting" as it were. We get

$$= e^{(-\gamma+i\omega)t}\int_0^t \underbrace{\left\{\int_u^t e^{-2i\omega v}\,dv\right\}}_{=\frac{e^{-2i\omega u}-e^{-2i\omega t}}{2i\omega}} e^{(\gamma+i\omega)u}S(u)\,du$$

which after the algebraic dust has settled becomes

$$x(t) = \tfrac{1}{\omega}\int_0^t e^{-\gamma(t-u)}\sin\omega(t-u)S(u)\,du$$

—in precise agreement with (25). As we have many times had occasion to remark, $\omega \equiv \sqrt{\omega_0^2-\gamma^2}$ is real, zero or imaginary according as γ^2 is less than, equal to or greater than ω_0^2. The following equations provide a manifestly real formulation of the result just obtained:[10]

$$x(t) = \begin{cases} \tfrac{1}{\omega}\displaystyle\int_0^t e^{-\gamma(t-u)}\sin\omega(t-u)S(u)\,du & : \text{ underdamped} \\ \displaystyle\int_0^t e^{-\gamma(t-u)}(t-u)S(u)\,du & : \text{ critically damped} \\ \tfrac{1}{\alpha}\displaystyle\int_0^t e^{-\gamma(t-u)}\sinh\alpha(t-u)S(u)\,du & : \text{ overdamped} \end{cases} \tag{27}$$

My remarks concerning the **interpretation** of (27) will hinge on properties of the **Dirac δ-function**, which occasion the following

> **DIGRESSION:** Shown in the FIGURE 12 is a sequence of "box functions"
>
> $$B(x-a;\epsilon) = \begin{cases} \frac{1}{2\epsilon} & : \quad a-\epsilon < x < a+\epsilon \\ 0 & : \quad \text{elsewhere} \end{cases}$$
>
> It becomes evident on a moment's thought that
>
> $$\lim_{\epsilon\downarrow 0}\int_{-\infty}^{+\infty} f(x)B(x-a;\epsilon)\,dx = f(a)$$
>
> Many other such sequences of functions would achieve the same result. The **Dirac δ-function**—casually said to

[10] Here I have, in the overdamped case, written $\alpha \equiv \sqrt{\gamma^2-\omega_0^2}$ and made use of the identity $\sin(i\theta) = i\sinh(\theta)$

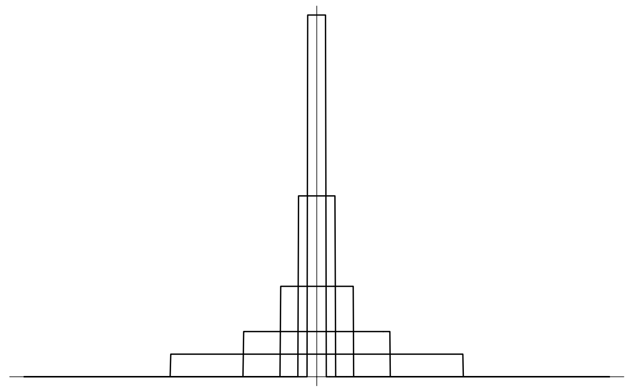


FIGURE 12: *Box functions get narrower but taller as $\epsilon \downarrow 0$, in such a way that the area under each remains unity.*

have the properties

$$\delta(x-a) = \begin{cases} 0 & : \quad x \neq a \\ \infty & : \quad x = a \end{cases}$$

$$\int \delta(x-a)\,dx = 0$$

though there exists no such "function." The "δ-function" is really just a notational device contrived to permit us by writing

$$f(a) = \int f(x)\delta(x-a)\,dx$$

to refer in a unified way to the limit of all such sequences of integrals. As such, it is a device that lives always "in the shade of an integral sign, whether explicit or implied. Without loss of generality we can/will assume $\delta(x)$ to be symmetric

$$\delta(x) = \delta(-x)$$

and—this is the main point—can consider

$$f(x) = \int f(a)\delta(x-a)\,da \tag{28}$$

to display $f(x)$ as a **weighted sum of unit spikes**, the spike at $x = a$ being weighted by the value $f(a)$ that the function f assumes there.

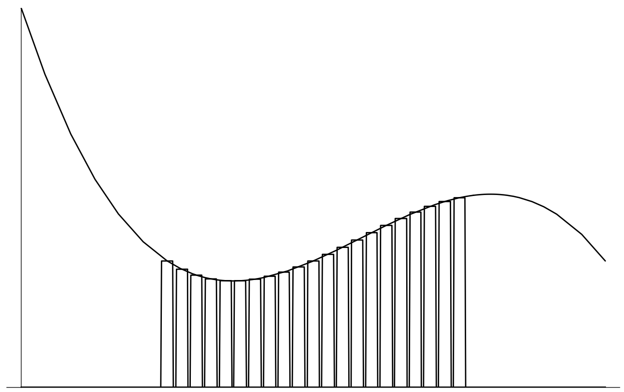


FIGURE 13: *Attempt to represent the upshot of* (28), *with kicks that are of finite width and height. Viewed in this light, Dirac's idea is seen to be a symbolic refinement of the construction that serves to* define *the Riemann integral.*

Returning now to (27)—which I will discuss in language specific to the underdamped case—we have

$$x(t) = \tfrac{1}{\omega}\int_0^t e^{-\gamma(t-u)}\sin\omega(t-u)S(u)\,du$$

$$S(u) = \int_{-\infty}^{+\infty} S(t')\delta(t'-u)\,dt'$$

where the lower limit on $\int_0^t$ reflects our former preoccupation with the initial conditions $x(0) = \dot{x}(0) = 0$. We have now no reason to preserve (and some reason to abandon) that preoccupation : writing $\int_{-\infty}^t$ in place of $\int_0^t$, we reverse the order of the integrals and obtain

$$x(t) = \int_{-\infty}^{+\infty} \underbrace{\Big\{\tfrac{1}{\omega}\int_{-\infty}^{t} e^{-\gamma(t-u)}\sin\omega(t-u)\delta(u-t')\,du\Big\}}_{\equiv\, G(t-t')} S(t')dt'$$

with

$$G(t-t') = \begin{cases} \tfrac{1}{\omega}e^{-\gamma(t-t')}\sin\omega(t-t') & : \quad t' < t, \text{ kick lies in the past} \\ \\ 0 & : \quad t' > t, \text{ kick lies in the future} \end{cases}$$

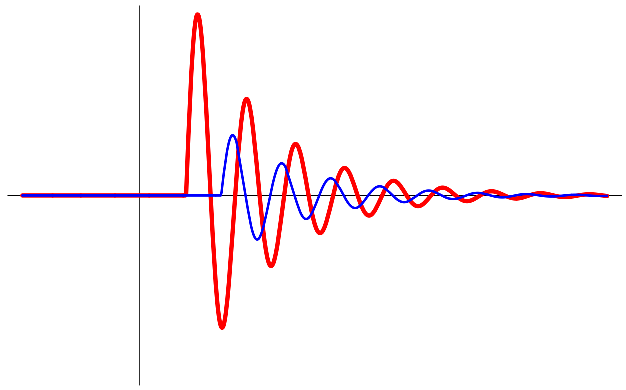


FIGURE 14: *"Green's function" $G(t-t')$ describes the response of the damped oscillator to a "unit kick" at time t', the assumption being that prior to the kick the oscillator was not moving. In the figure the oscillator is kicked, then a bit later kicked again, but more weakly. We consider $S(t')$ to describe a continuous stream of kicks of graded intensity.*

We conclude that the response of a damped oscillator to the stimulus

$$S(t) = \int_{-\infty}^{+\infty} \delta(t-t')S(t')\,dt' \tag{29.1}$$

can be described

$$x(t) = \int_{-\infty}^{t} G(t-t')S(t')\,dt' \tag{29.2}$$

and that this can be considered to have resulted by superposition from

$$\big[D^2 + 2\gamma D + \omega_0^2\big]G(t) = \delta(t) \tag{30}$$

The truncated integral in (29.2) reflects the physicists' conception of

CAUSALITY: Output does not precede input,
response does not precede stimulus.

PROBLEM 13: a) The "Heaviside step function" is defined

$$\theta(x) \equiv \begin{cases} 0 & : \quad x<0 \\ \frac{1}{2} & : \quad x=0 \\ 1 & : \quad x>0 \end{cases}$$

and is known to *Mathematica* as `UnitStep[x]`. Plot $\theta(x)$ on the interval $-5 \leqslant x \leqslant +5$.

b) What does *Mathematica* have to say about $\frac{d}{dx}\theta(x)$?

c) Evaluate $\int_{-\infty}^{x}\delta(y)\,dy$.

d) Construct

$$G(t) \equiv \frac{e^{-\gamma t}\sin\sqrt{\omega_0^2-\gamma^2}\,t}{\sqrt{\omega_0^2-\gamma^2}}\cdot\theta(t)$$

What does *Mathematica* have to say about $\big[D^2+2\gamma D+\omega_0^2\big]G(t)$?

Equations (29) provide one concrete realization of an elementary idea

$$\left.\begin{array}{c}\textbf{general stimulus} = \textbf{weighted sum of simple stimuli}\\ \Downarrow\\ \textbf{response} = \textbf{identically weighted sum of simple responses}\end{array}\right\}\quad(31)$$

that can be implemented also in other ways. Suppose, for example, we were to write[11]

$$S(t) = \text{real part of} \int_0^\infty e^{i\nu t}\cdot\sigma(\nu)\,d\nu \tag{32.1}$$

From results obtained on page 15 it would then follow that

$$x(t) = \text{real part of} \int_0^\infty A(\nu)e^{i[\nu t-\delta(\nu)]}\cdot\sigma(\nu)\,d\nu \tag{32.2}$$

We begin to gain some respect for the engineers who design acoustic microphones and speakers, or who have to contend with just about *any* kind of vibrating structure, whether the vibrations be intentional (as in a musical instrument) or unintentional (seismic shaking of a building, or of a bridge by the tramp, tramp, tramp of Roman soldiers on the march). A speaker that responds to a handclap with a WAA-waa-waa, a microphone that is sensitive only to a narrow range of the Fourier components present in the input signal (those with $\nu\sim\omega$)—such designs would be considered to be in need of further work. The short of it: all stimulated vibrating objects can be expected to exhibit **resonance phenomena**, which in some applications will present themselves as a "problem to be solved," but in others (vibration of molecules, of stars) become our principal source of structural information.

6. Electrical analogs of mechanical oscillators. The theory of oscillations derives its importance partly from the circumstance that it pertains simultaneously to objects/systems of so many different types. A prominent area of application of

[11] We owe to Fourier the discovery that essentially *any* $S(t)$ can be developed in this way: **Fourier analysis** and the **theory of the Fourier transform** are subjects to which we will return.

the theory developed in the preceding section is to man-made electronic devices of the sorts indicated in the following figures. Familiarly, the

$$\begin{aligned}
\text{potential drop across an inductor} &= L\cdot \tfrac{d}{dt}(\text{current}) &&= L\cdot \tfrac{d^2}{dt^2}(\text{charge})\\
\text{potential drop across a resistor} &= R\cdot(\text{current}) &&= R\cdot \tfrac{d}{dt}(\text{charge})\\
\text{potential drop across a capacitor} &= (\text{charge})/\text{C}
\end{aligned}$$

Looking first to the system diagrammed in FIGURE 15, we assume the capacitor to have carried an initial charge q_0 and the switch to have been closed at time $t=0$. At subsequent times we (by Kirchhoff's 2$^{\text{nd}}$ Law) have

$$L\ddot{q} + R\dot{q} + C^{-1}q = 0, \quad q(0) = q_0, \quad q(0) = 0$$

These equations provide an instance of (21.1) with

$$2\gamma = R/L, \quad \omega_0^2 = 1/LC$$

Energy dissipation is attributable in this instance to the phenomenon of electrical resistance: not to radiation of any kind, but to I^2R *heating*.

Temporal behaviour of the externally driven system diagrammed in FIGURE 15 can be described

$$L\ddot{q} + R\dot{q} + C^{-1}q = V(t) \tag{33}$$

which provides an instance of (21.2) with

$$2\gamma = R/L, \quad \omega_0^2 = 1/LC, \quad S(t) = V(t)/L$$

More complicated systems of coupled RLC circuits will (not at all surprisingly) turn out to be formally identical to more complicated systems of quivering particles.

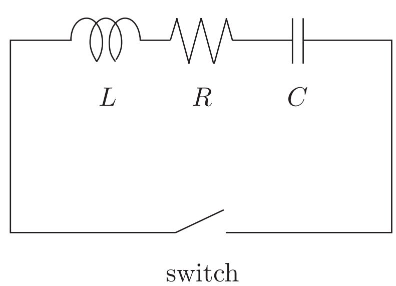


FIGURE 15: *The capacitor is charged up, the switch is closed, the charge sloshes back and forth (or not, depending upon the sign of $4L/C - R^2$) until the energy initially stored in the capacitor has been dissipated by the resistor.*

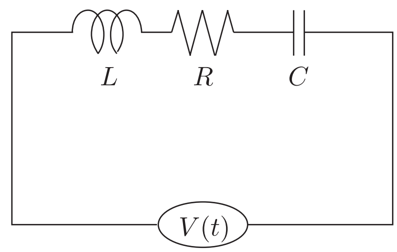


FIGURE 16: *Simple RLC circuit in which charge motion (current) is driven by an eternally powered signal generator.*

Often it proves useful—as an aid to thought, but also in the laboratory—to work with the electrical analogs of mechanical oscillatory devices, rather than with the mechanical devices themselves. . . as we will have occasion to do.

7. Physics of swinging. It is a fact—known to every "big kid" in the park—that it is entirely possible to swing without the assistance of a parent, a "pusher;" possible to feed energy into an oscillator without the application of an externally impressed (and appropriately synchronized) force $F(t)$. How is this done?

To begin at the beginning: a particle m is imagined to move subject to forces of three types

$$\begin{aligned}\tfrac{d}{dt}(\text{momentum}) &= \text{spring force} \ + \ \text{drag} \ + \ \text{impressed force}\\ \tfrac{d}{dt}(m\dot{x}) &= -kx - b\dot{x} + F(t)\end{aligned}$$

and with energy

$$\begin{aligned}E &= \text{kinetic energy} \ + \ \text{potential energy}\\ &= \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2\end{aligned}$$

Arguing from

$$\dot{E} = (m\ddot{x} + kx)\dot{x}$$

and the equation of motion

$$m\ddot{x} + kx = -b\dot{x} + F(t)$$

one obtains

$$\dot{E} = -b\dot{x}^2 + F(t)\dot{x}$$

and concludes that energy will be conserved if and only if the terms on the right conspire to vanish. One expects on physical grounds to have $b \geqslant 0$ so the leading term on the right (the drag term) always leads to energy dissipation, but $F(t)\dot{x}$ can have either sign, can either inject energy into or extract energy from the oscillator.

Here it has been tacitly assumed that the parameters m, b and k are constants. If, however, they are subject to external manipulation then

$$\dot{E} = (m\ddot{x} + kx)\dot{x} + \tfrac{1}{2}\dot{m}\dot{x}^2 + \tfrac{1}{2}\dot{k}x^2$$

and the equation of motion supplies

$$m\ddot{x} + kx = -\dot{m}\dot{x} - b\dot{x} + F(t)$$

giving

$$\dot{E} = -(b + \tfrac{1}{2}\dot{m})\dot{x}^2 + \tfrac{1}{2}\dot{k}x^2 + F(t)\dot{x} \tag{34}$$

Now *each of the terms on the right can assume either sign*. It has become **possible by parametric manipulation to tickle energy into the system**, even in the absence of impressed forces $F(t)$.

Look in this light to the pendulum shown in FIGURE 17, which—as indicated—is equivalent to an unforced oscillator with the ℓ-dependent spring constant

$$k = mg/\ell$$

Equation (34) in this instance becomes

$$\begin{aligned}\dot{E} &= \tfrac{1}{2}\dot{k}x^2 - b\dot{x}^2\\ &= -\tfrac{1}{2}mg(x/\ell)^2\dot{\ell} - b\dot{x}^2\end{aligned}$$

The second term on the right is always non-negative, but the first term is positive or negative according as $\dot{\ell}$ is negative or positive. The implication is that to inject energy into such an oscillator—to "pump" such a swing— the controlling agent should

- make $\dot{\ell} < 0$ (be in process shortening the string, which will require doing some work) when x is extremal;
- wait until x is small again before restoring ℓ to its original value (which will entail allowing $\dot{\ell}$ to go positive).

Which—as every child knows—is how one *does* pump a swing (not actually by manipulating the length of the support cable but by manipulating the distance from the pivot point to one's center of mass, which amounts to the same thing).

Suppose we were to set $\ell(t) = \ell_0[1+\kappa\cos 2\omega_0 t]$, where the doubled frequency finds its motivation in FIGURE 18 and where $\omega_0 = \sqrt{g/\ell_0}$. Then

$$k = \frac{mg}{\ell_0[1 + 2\kappa\cos 2\omega_0 t]} = m\omega_0^2\big\{1 - 2\kappa\cos 2\omega_0 t + \cdots\big\}$$

and—for κ sufficiently small—the equation of motion becomes

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2[1 - 2\kappa\cos 2\omega_0 t]x = 0 \tag{35}$$

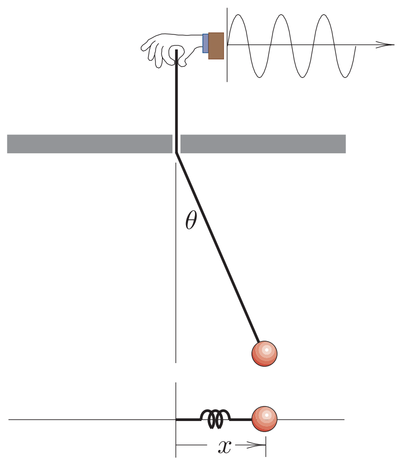


FIGURE 17: *Shown above*: *a simple pendulum (or "swing") in which the length of the support cable is subject to manipulation by an external agent. Below*: *the equivalent system in which the spring constant has become subject to time-dependent variation.*

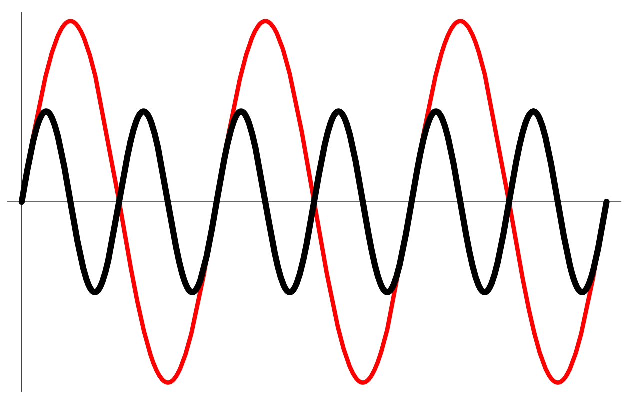


FIGURE 18: *The red curve depicts the undisturbed motion of the oscillator. The black curve has been designed to be rising whenever the red curve is extremal, descending whenever the red curve is passing through zero. The curve explains the origin of the* **frequency doubling** *that is characteristic of parametric stimulation.*

We have in (35) encountered an instance of **Mathieu's equation**

$$\ddot{x} + 2\gamma\dot{x} + (a - b\cos 2t)x = 0$$

the general solution of which is (use *Mathematica*'s `DSolve` command)

$$x(t) = e^{-\gamma t}\Big\{X_1\, \texttt{MethieuC}\Big[a-\gamma^2,\, b,\, t\Big] + X_2\, \texttt{MethieuS}\Big[a-\gamma^2,\, b,\, t\Big]\Big\}$$

where `MethieuC` and `MethieuS` are **Mathieu functions**. Mathieu's equation is encountered quite commonly in the theory of parametrically stimulated oscillators, but it was theory pertaining to the vibration of elliptical membranes that motivated Emile Mathieu (1835–1900) to take up the subject. The theory of Mathieu functions[12] is exceptionally (!) intricate, but a few simple things can be said... among them, that `MathieuC[a,b,t]` is an *even* function, and `MathieuS[a,b,t]` an odd function of `t`. Imposition of the initial condition $\dot{x}(0) = 0$ would therefore force us to set $X_2 = 0$. If we require further that $x(0) = x_0$ then we have

$$x(t) = x_0 \frac{e^{-\gamma t}\,\texttt{MathieuC[}\,a-\gamma^2,\, b,\, t\,\texttt{]}}{\texttt{MathieuC[}\,a-\gamma^2,\, b, 0\,\texttt{]}}$$

It is useful in this connection to notice also that if we set b (which is in effect to turn off the parametric stimulation) then Mathieu's equation (whence also its solutions, the Mathieu functions) becomes a much less fearsome beast:

$$e^{-\gamma t}\,\texttt{MathieuC[}\,a-\gamma^2,\, 0,\, t\,\texttt{]} = e^{-\gamma t}\cos\big[\,t\sqrt{a-\gamma^2}\,\big]$$

Reinstating the physical parameters present in (35) we (with *Mathematica*'s assistance) obtain

$$x(t;\omega_0) = x_0 \frac{e^{-\gamma t}\,\texttt{MathieuC[}\,(\omega/\omega_0)^2,\, \kappa,\, \omega_0 t\,\texttt{]}}{\texttt{MathieuC[}\,(\omega/\omega_0)^2,\, \kappa, 0\,\texttt{]}} \tag{36}$$

with $\omega \equiv \sqrt{\omega_0^2 - \gamma^2}$. Figures based upon (36) ae shown on the next page. We are in position now to compute, if we were so inclined, the *energy injected per period*

$$\Delta E \equiv \int_t^{t+\tau} \dot{E}(t')\,dt'$$

and the *mean energy*

$$\bar{E} \equiv \tfrac{1}{\tau}\int_t^{t+\tau} E(t')\,dt'$$

and in $J \equiv \Delta E/\bar{E}$ possess a plausible measure of the **efficiency** of the pumping process.

[12] See M. Abramowitz & I. Stegun, *Handbook of Mathematical Functions* (1964), Chapter 20; *Encyclopedic Dictionary of Mathematics* (2$^{\text{nd}}$ edition 1987), §268; http://mathworld.wolfram.com/MathieuFunction.html.

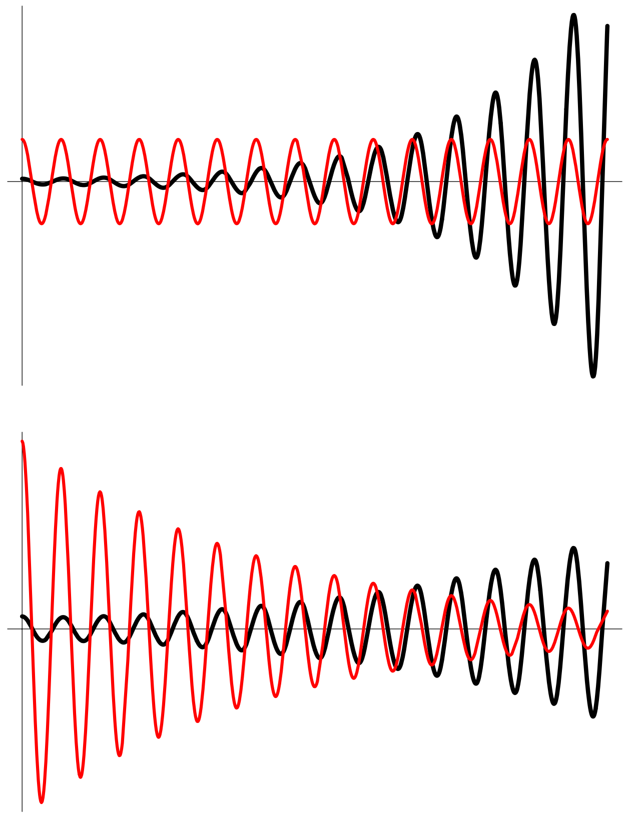


FIGURE 19: *At top, the black curve is a graph of the stimulated but undamped motion* $x(t;1,0,\frac{1}{10})$ *while the red curve—drawn for purposes of comparison—is a graph of the unstimulated undamped motion of the same period:* $x(t;1,0,0)$*. At bottom, both the period and the strength of the parametric stimulation have remained the same, but damping of strength* $\gamma = 0.025$ *has been introduced. This rusty swing is harder to pump.*

The "theory of swinging" developed above is a little simplistic, for it takes no account of the fact when we swing we seldom just stand in the seat and flex our knees: we find it natural (and more efficient), whether sitting or standing, to rock back and forth in synchrony with the swing's motion. W. B. Case has looked carefully into the physics of the matter, and shown it to be much more interesting than one might have anticipated.[13]

It is reported to have been Faraday who, working with a fluid-dynamical system, first observed (1831) the phenomenon of parametric resonance, and

[13] "The pumping of a swing from the standing position," AJP **64**, 215 (1996). I am indebted to David Griffiths for bringing to my attention this paper, which provides also some additional references.

it was the vibration of fluids and elastic structures that inspired most of the experimental/theoretical work in the area for the better part of a century.[14] In 1934 L. Mandelstam & N. Papalexi constructed a "parametric generator" and found that "if the circuit of the parametric generator is linear the amplitude of the oscillation grows indefinitely until the insulation [on the wires melts]," but that such catastrophy "can be avoided by the introduction of nonlinear circuit elements."[15] It is my impression that Mandelstam & Papalexi used a circuit

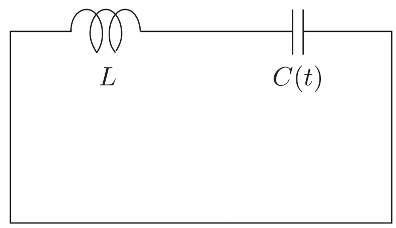


FIGURE 20: *This simple LC circuit would become a parametric oscillator if the plate separation in the capacitor were made—by means of a piezoelectric crystal, let us say—to oscillate about some mean value. Alternatively (John Essick's suggestion) one might dither the core of the inductor.*

rather like that shown above, the theory of which—if one sets

$$C(t) = C_0[1 + 2\kappa\cos 2\omega_0 t] \quad \text{with} \quad \omega_0 \equiv 1/\sqrt{LC_0}$$

—is readily seen to be formally identical to the previously-discussed theory of pumped swings.

Parametrically stimulated devices have in recent times become very powerful tools in the hands of researchers (such as our own Danielle Braje!) who use nonlinear optical effects to study properties of atoms. Laser light is projected into an atomic vapor, a second laser is used to modulate certain system parameters, and information is obtained by analysis of the emergent light. For further information ask Google for sources that treat "parametric amplification," "parametric oscillation," *etc*. Or ask Danielle!

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[14] L. Brillouin did, however, study parametric resonance in electric circuits as early as 1897. For a good brief history of the theory of parametric resonance see the introduction to Chapter 5 ("Parametrically Excited Systems") in Ali Hasan Nayfeh & Dean T. Mook, *Nonlinear Oscillations* (1979).

[15] Recall in this connection that the solutions of $\dot{x} = ax$ grow indefinitely (exponentially!) if $a > 0$, while the solutions of the nonlinear "logistic equation" $\dot{x} = a(b - x)x$ are all stably asymptotic to b: $\lim_{t\to\infty} x(t) = b$.

To summarize: systems of type

$$m\ddot{x} + b\dot{x} + kx = F(t) \tag{37.1}$$

are said to be "forced" (or externally driven), while systems of type

$$m(t)\ddot{x} + b(t)\dot{x} + k(t)x = 0 \tag{37.2}$$

are said to be "parametrically excited." For systems of both types (it is, of course, possible to construct systems that combine *both* types of stimulation) **energy is generally not conserved** (is injected/extracted). Efficient injection into systems of type (37.1) requires that predominant forcing frequencies be tuned to the natural frequency of the oscillator, while efficient injection into systems of type (37.2) that the precominant frequency of parametric variation be tuned to <u>twice</u> the natural frequency of the oscillator.